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PERIODIC SOLUTIONS FOR A PRESCRIBED-ENERGY PROBLEM OF SINGULAR HAMILTONIAN SYSTEMS

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Abstract. We study the existence of periodic solutions for a prescribed-energy problem of Hamiltonian systems whose potential function has a singularity at the origin like $-1/|q|^\alpha (q \in \mathbb{R}^N)$. It is known that there exist generalized periodic solutions which may have collisions, and the number of possible collisions has been estimated. In this paper we obtain a new estimation of the number of collisions. Especially we show that the obtained solutions have no collision if $N \geq 2$ and $\alpha > 1$.

1. **Introduction and Main Theorem.** We consider an autonomous Hamiltonian $H : \mathcal{D}(\subset \mathbb{R}^{2N}) \to \mathbb{R}$. Along each solution of the Hamiltonian system, H is conserved. Then each solution is on the energy surface

$$S_h = \{x \in \mathcal{D} \mid H(x) = h\}$$

for some h.

The existence problem of a periodic solution on S_h for a prescribed energy h has been studied vigorously, especially in the case that S_h is compact. The existence has been proved for the compact convex energy surfaces [10], the compact star-shaped surfaces [7] and the compact contact manifolds [3, 9].

As a noncompact and physically important case, Ambrosetti and Coti-Zelati [2] and Tanaka [8] have studied the natural Hamiltonian system

$$H = \frac{1}{2}\left|\frac{dq}{dt}\right|^2 + V(q) \qquad (q \in \mathbb{R}^N).$$

whose potential $V(q) \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$ behaves like $-\frac{1}{|q|^\alpha}$ near 0. They have proved the existence of a generalized solution (a solution possibly with collisions) by attaining a minimax point for the functional

$$\mathcal{I}_h(q) = \frac{1}{2}\int_0^1 \left|\frac{dq}{d\tau}\right| d\tau \int_0^1 h - V(q) d\tau, \tag{1}$$

and has estimated the number of collisions. Tanaka has proved that the obtained solution is collisionless if the potential $V(q)$ is in a class of functions including

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$-1/|q|^\alpha$ for

$$\frac{4}{3} < \alpha < 2 \qquad (N = 3)$$
$$1 < \alpha < 2 \qquad (N \geq 4).$$

We will provide the existence of non-collision periodic solution for a class including $-1/|q|^\alpha$ for

$$1 < \alpha < 2 \qquad (N \geq 2).$$

The functional (1) is regarded as the functional with respect to geodesics on some Riemannian manifold. Remark that the Riemannian metric is degenerate on the boundary

$$\{q \in \mathbb{R}^N \mid U(q) = h\}.$$

For example, Keplerian potential $V(q) = -1/|q|$ can be represented by geodesics on the surface with constant curvature(see [5]). In the three-body problem with strong force, it is proven that the dynamics is hyperbolic by showing that the curvature is negative(see [6]). Similarly triple linkage system is shown to be Anosov (see [4]).

Consider the natural Hamiltonian system

$$H = \frac{1}{2}\left|\frac{dq}{dt}\right|^2 + V(q) \qquad (p \in \mathbb{R}^N, q \in \mathbb{R}^N \backslash \{0\}). \tag{2}$$

with a potential $V(q) \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$. The canonical equations are represented by

$$\frac{d^2q}{dt^2} = -\nabla V(q). \tag{3}$$

Let h be a real number. Consider a periodic solution having the energy h:

$$\frac{1}{2}\left|\frac{dq}{dt}\right|^2 + V(q) = h. \tag{4}$$

Definition 1.1. We call $q(t)$ a generalized periodic solution of (3) and (4) with period T if

1. $q \in C(\mathbb{R}, \mathbb{R}^N)$ and T-periodic,
2. $D = \{t \in \mathbb{R} \mid q(t) = 0\}$ has zero measure,
3. $q \in C^2(\mathbb{R} \backslash D, \mathbb{R}^N)$ satisfies (3) and (4) in $\mathbb{R} \backslash D$

Theorem 1.2. *([8, Theorem 0.2]) Assume*

1. *$V(q) \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$;*
2. *$V(q) < 0$ for all $q \in \mathbb{R}^N \backslash \{0\}$ and $V(q), \nabla V(q) \to 0$ as $|q| \to \infty$;*
3. *There is an $\alpha_1 \in (0, 2)$ such that*
$$\nabla V(q)q \geq -\alpha_1 V(q)$$
for all $q \in \mathbb{R}^N \backslash \{0\}$;
4. $$V(q) = -\frac{1}{|q|^\alpha} + W(q)$$
where $\alpha \in (0, 2)$ and $|q|^\alpha W(q), |q|^{\alpha+1} \nabla W(q), |q|^{\alpha+2} \nabla^2 W(q) \to 0$ as $|q| \to 0$,

and $h < 0$. Then

- **(i):** *in case $N \geq 4$, we have*
  - **(a):** *if $\alpha \in (1, 2)$, then (2) and (4) possesses at least one classical solution;*

(b): *if* $\alpha \in (0,1]$, *then* (2) *and* (4) *possesses a generalized solution which enters the singularity* 0 *at most one time in its period.*

(ii): *in case* $N = 3$, *we have*

(a): *if* $\alpha \in (4/3, 2)$, *then* (2) *and* (4) *possesses at least one classical solution;*

(b): *if* $\alpha \in (1, 4/3]$, *then* (2) *and* (4) *possesses a generalized solution which enters the singularity* 0 *at most one time in its period;*

(c): *if* $\alpha \in (0,1]$, *then* (2) *and* (4) *possesses a generalized solution which enters the singularity* 0 *at most two times in its period.*

Our main theorem is the following:

Theorem 1.3. *Let* $N \geq 2$ *and* $V \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$. *Assume that for* $0 < a_1 < a_2, 0 < \alpha_1 < \alpha < \alpha_2 < 2$, $V(q)$ *satisfies*

$$\frac{a_1}{|q|^\alpha} \leq -V(q) \leq \frac{a_2}{|q|^\alpha} \tag{5}$$

$$-\alpha_1 V(q) \leq \nabla V(q) \cdot q \leq -\alpha_2 V(q) \tag{6}$$

$$\nabla V(q) \to 0 \text{ as } |q| \to \infty \tag{7}$$

$$|q|^3 \nabla V(q), |q|^4 \nabla^2 V(q) \to 0 \text{ as } q \to 0. \tag{8}$$

Then for any $h < 0$, *there is a generalized periodic solution of* (3) *with* (4). *Let* $T > 0$ *be the minimal period. The number of collisions is estimated as follows:*

$$\#\{t \in [0, T) \mid q(t) = 0\} \leq f\left(\frac{a_1}{a_2}, \alpha, \alpha_1, \alpha_2\right).$$

Here f *is defined by*

$$f(b, \alpha, \alpha_1, \alpha_2) = \frac{\pi b^{\frac{1}{\alpha}} \alpha^{\frac{3}{2}} (2-\alpha)^{\frac{2}{\alpha}} (2+\alpha_2)^{\frac{2+\alpha}{2\alpha}}}{2^{\frac{1}{\alpha}} \alpha_1 (2+\alpha)^{\frac{2+\alpha}{2\alpha}} (2-\alpha_2)^{\frac{2-\alpha}{2\alpha}} B\left(\frac{1}{2}, \frac{2+\alpha}{2\alpha}\right)}$$

and B *is the beta function:*

$$B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds.$$

Note that the function $U_\alpha(q) = -\frac{1}{|q|^\alpha}$ satisfies

$$\nabla U_\alpha(q) \cdot q = -\alpha U_\alpha(q).$$

Therefore, the inequality (6) is a condition weakening the equality. Then $U_\alpha(q) = -\frac{1}{|q|^\alpha}$ satisfies the assumptions of this theorem.

Figure 1 stands for the graph of $f(1, \alpha, \alpha, \alpha)$. We get the following corollaries.

Corollary 1. *For any* $\alpha \in (1, 2)$, *there is a* $\delta > 0$ *small enough such that if* $a_1, a_2, \alpha_1, \alpha_2$ *satifies* $a_1 \leq a_2 < (1+\delta)a_1, 0 < \alpha_2 - \alpha_1 < \delta$, *and the potential* V *satisfies the same properties as Theorem 1.3, then the obtained solution has no collision, and hence is a classical solution.*

Corollary 2. *For any* $\alpha \in (0, 1)$, *there is a* $\delta > 0$ *small enough such that if* $a_1, a_2, \alpha_1, \alpha_2$ *satifies* $a_1 \leq a_2 < (1+\delta)a_1, 0 < \alpha_2 - \alpha_1 < \delta$, *and the potential* V *satisfies the same properties as Theorem 1.3, then the obtained solution enters the singularity* 0 *at most one time in its period.*

Our proof is based on a "global estimate". Such a research has been done (see [1, Section 14]). But our result is stronger than the existing results. We get it by making a technical refinement.

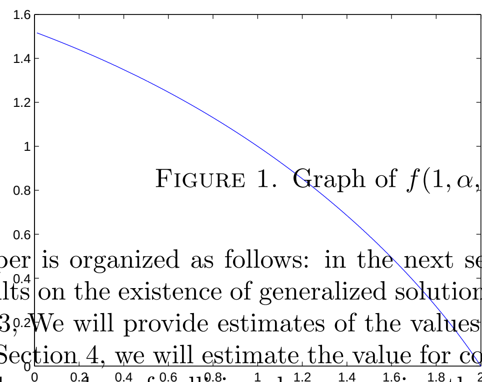

FIGURE 1. Graph of $f(1, \alpha, \alpha, \alpha)$.

This paper is organized as follows: in the next section, we will introduce some known results on the existence of generalized solutions obtained as minimax points. In Section 3, We will provide estimates of the values of the functional for minimax points. In Section 4, we will estimate the value for collision paths and obtain upper bound of the number of collisions by comparing those values. In the last section, we will show the corollaries.

2. **Existence of a generalized solution.** The prescribed-energy problem is represented by the variational problem with respect to the functional

$$\mathcal{I}_h(u) = \frac{1}{2}\int_0^1 \left|\frac{du}{d\tau}\right|^2 d\tau \int_0^1 h - V(u(\tau))d\tau.$$

For a critical point $u(\tau)$ of $\mathcal{I}_h$, $q(t) = u(t/T)$ is a solution of (3) and (4) where

$$T = \sqrt{\frac{\frac{1}{2}\int_0^1 \left|\frac{du}{d\tau}\right|^2 d\tau}{\int_0^1 h - V(u)d\tau}}. \tag{9}$$

We first consider the case of $N \geq 3$. The existence of a generalized solution has been proved by Ambrosetti and Coti-Zelati [2] and Tanaka [8]. We survey the result here according to [8]. Let

$$E = \{u(\tau) \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N); u(\tau+1) = u(\tau)\}$$
$$\Lambda = \{u(\tau) \in E; u(\tau) \neq 0 \text{ for all } \tau\}.$$

We set Λ as the domain of $\mathcal{I}_h$. Define a map $\sigma : S^{N-2} \times S^1 \to \mathbb{R}^N$ by

$$\sigma(\xi, \tau) = ((2 + \cos 2\pi\tau)\xi_1 + 2, (2 + \cos 2\pi\tau)\xi_2, \ldots, (2 + \cos 2\pi\tau)\xi_{N-1}, \sin 2\pi\tau),$$

and a family of maps

$$\Gamma = \{\gamma(r, x) \in C([0, 1] \times S^{N-2}, \Lambda); \gamma(0, x)(\tau) = \underline{R}\sigma(x, \tau), \gamma(1, x)(\tau) = \bar{R}\sigma(x, \tau)\}$$

where $\underline{R} > 0$ and $\bar{R} > 0$ are sufficiently small and large numbers respectively. The generalized solution can be obtained by attaining a minimax value:

$$b = \inf_{\gamma \in \Gamma} \max_{(r,x) \in [0,1] \times S^{N-2}} \mathcal{I}_h(\gamma(r, x)).$$

Theorem 2.1. *([8, Theorem 0.1]) Assume $N \geq 3$ and*

1. $V(q) \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$*;*
2. $V(q) < 0$ *for all* $q \in \mathbb{R}^N \backslash \{0\}$ *and* $V(q), \nabla V(q) \to 0$ *as* $|q| \to \infty$*;*
3. *There is an* $\alpha_1 \in (0,2)$ *such that*
$$\nabla V(q)q \geq -\alpha_1 V(q)$$
for all $q \in \mathbb{R}^N \backslash \{0\}$*;*
4. *There is an* $\alpha_2 \in (0,2)$ *and* R_0 *such that*
$$\nabla V(q)q \leq -\alpha_2 V(q)$$
for all $0 < |q| \leq R_0$*;*
5. $|q|^3 \nabla V(q), |q|^4 \nabla^2 V(q) \to 0$ *as* $q \to 0$*.*

Then for any $h < 0$*, there exists a generalized solution* $q(t)$ *attaining* b*.*

Remark 1. In [8], the case $N = 2$ was not written because the estimate of the Morse index for collision paths cannot be obtained. But the existence of a generalized solution can be shown more easily as follows: we consider $\rho_R(\tau)$ defined by
$$\rho_R(\tau) = R(\cos 2\pi\tau, \sin 2\pi\tau).$$
Take small $R_0 > 0$ and large $R_1 > 0$. Let
$$Q = \{\eta \in C([R_0, R_1], \Lambda) \mid \eta(R_0) = \rho_{R_0}, \eta(R_1) = \rho_{R_1}\}.$$
We can get a generalized solution attaining
$$c = \inf_{\eta \in Q} \max_{R \in [R_0, R_1]} I(\eta(R)).$$

Under the assumptions in Theorem 1.3, this theorem indicates the existence of a generalized solution. A problem is whether the obtained solution has a collision, or how many collisions the solution has. Tanaka [8] also has estimated the number of collisions which the generalized solution has by computing the Morse index to show Theorem 1.2.

3. **Estimate of the minimax value.** To prove our theorem, we estimate the minimax value here. We start with the case $N = 2$. We estimate the value of $\mathcal{I}$ for the curve ρ_R in Remark 1. From the inequality (5), we get
$$\begin{aligned} I(\rho_R) &= \frac{1}{2}\int_0^1 \left|\frac{d\rho_R}{d\tau}\right|^2 d\tau \int_0^1 h - V(\rho_R) d\tau \\ &\leq \frac{1}{2}\int_0^1 \left|\frac{d\rho_R}{d\tau}\right|^2 d\tau \int_0^1 h + \frac{a_2}{|\rho_R|^\alpha} d\tau \\ &= 2\pi^2(hR^2 + a_2 R^{2-\alpha}). \end{aligned}$$
The maximum of $2\pi^2(hR^2 + a_2R^{2-\alpha})$ on $R \in [R_0, R_1]$ is
$$\pi^2 \alpha a_2^{\frac{2}{\alpha}} \left(\frac{2-\alpha}{-2h}\right)^{\frac{2-\alpha}{\alpha}}.$$
Therefore, the minimax values is no more than this value:
$$c \leq \pi^2 \alpha a_2^{\frac{2}{\alpha}} \left(\frac{2-\alpha}{-2h}\right)^{\frac{2-\alpha}{\alpha}}. \tag{10}$$

For the case of $N \geq 3$, we take σ defined in the previous section. It is easy to show

$$|\sigma| \geq 1$$

and

$$\left|\frac{d\sigma}{d\tau}\right| = 1.$$

Therefore, we have the same estimate as the case $N = 2$:

$$I(R\sigma(\xi,\cdot)) \leq 2\pi^2(hR^2 + a_2R^{2-\alpha}),$$

and hence

$$b \leq \pi^2\alpha a_2^{\frac{2}{\alpha}}\left(\frac{2-\alpha}{-2h}\right)^{\frac{2-\alpha}{\alpha}}. \tag{11}$$

is held also for $N \geq 3$.

4. **Estimate for collision paths.** Here we estimate the value of the functional for a generalized solution with collisions k times per the period. Assume $\gamma(\tau)$ is a generalized solution with collisions at $0 \leq \tau_0 < \tau_1 < \cdots < \tau_{k-1} < 1$. We estimate the value of the integral on $[\tau_{i-1}, \tau_i]$ which appeared in the functional $\mathcal{I}_h$. The following proof works for the case $k = \infty$, and it is easy to show $\mathcal{I}_h = \infty$ if $k = \infty$. We can assume $\tau_0 = 0$ without loss of generality. Let T be the period defined by (9). Let $t = T\tau$ and $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = T$ be the corresponding collision times, i. e. $T_i = \tau_i T$, and let $T_k = T$.

By substituting t for τ, we have

$$\begin{aligned}
I(\gamma) &= \frac{1}{2}\int_0^1 \left|\frac{d\gamma}{d\tau}\right|^2 d\tau \int_0^1 h - V(\gamma)d\tau \\
&= \frac{1}{2}\int_0^T \left|\frac{d\gamma}{dt}\right|^2 dt \int_0^T h - V(\gamma)dt \\
&= \left(\frac{1}{2}\sum_{i=1}^{k}\int_{T_{i-1}}^{T_i} \left|\frac{d\gamma}{dt}\right|^2 dt\right)\left(\sum_{i=1}^{k}\int_{T_{i-1}}^{T_i} h - V(\gamma)dt\right)
\end{aligned}$$

Since the conservation law

$$\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 + V(\gamma) = h,$$

we get

$$\frac{1}{2}\int_{T_{i-1}}^{T_i} \left|\frac{d\gamma}{dt}\right|^2 dt = \int_{T_{i-1}}^{T_i} h - V(\gamma)dt.$$

We will estimate

$$\frac{1}{2}\int_0^{T_1} \left|\frac{d\gamma}{dt}\right|^2 dt \left(= \int_0^{T_1} h - V(\gamma)dt\right).$$

We have

$$\int_0^{T_1} \left|\frac{d\gamma}{dt}\right|^2 dt = \left[\gamma\cdot\frac{d\gamma}{dt}\right]_0^{T_1} - \int_0^{T_1} \gamma\cdot\frac{d^2\gamma}{dt^2}dt = \int_0^{T_1} \gamma\cdot\nabla V dt,$$

where we use

$$\begin{aligned}\left|\gamma(t)\cdot\frac{d\gamma}{dt}(t)\right| &\le |\gamma(t)|\left|\frac{d\gamma}{dt}(t)\right| = |\gamma(t)|\sqrt{2(h-V(\gamma(t)))}\\ &\le |\gamma(t)|\sqrt{2(h+a_2|\gamma(t)|^{-\alpha})} = \sqrt{2(h|\gamma(t)|^2+a_2|\gamma(t)|^{2-\alpha})}\to 0\end{aligned}\tag{12}$$

as $t\to +0$ or $t\to T_1-0$. From (6) we get

$$-\alpha_1\int_0^{T_1} V(\gamma)dt \le \int_0^{T_1}\left|\frac{d\gamma}{dt}\right|^2 dt \le -\alpha_2\int_0^{T_1} V(\gamma)dt.$$

Since

$$hT_1 = \int_0^{T_1} hdt = \int_0^{T_1}\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 + V(\gamma)dt,$$

the energy value is estimated as follows:

$$\frac{2-\alpha_1}{2}\int_0^{T_1} V(\gamma)dt \le hT_1 \le \frac{2-\alpha_2}{2}\int_0^{T_1} V(\gamma)dt.\tag{13}$$

Similarly we can estimate the action functional

$$\mathcal{A}_{T_1} = \int_0^{T_1}\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(\gamma)dt$$

as follows:

$$\frac{-2-\alpha_1}{2}\int_0^{T_1} V(\gamma)dt \le \mathcal{A}_{T_1} \le \frac{-2-\alpha_2}{2}\int_0^{T_1} V(\gamma)dt.\tag{14}$$

The inequalities (13) and (14) indicate

$$-hT_1 \ge -\frac{2-\alpha_2}{2}\int_0^{T_1} V(\gamma)dt \ge \frac{2-\alpha_2}{2+\alpha_2}\mathcal{A}_{T_1}.\tag{15}$$

The value we need to estimate is related to one of the action functionals:

$$\frac{1}{2}\int_0^{T_1}\left|\frac{d\gamma}{dt}\right|^2 dt = \int_0^{T_1} h - V(\gamma)dt \ge \frac{\alpha_1}{2}\int_0^{T_1} -V(\gamma)dt \ge \frac{\alpha_1}{2+\alpha_2}\mathcal{A}_{T_1}.$$

On the other hand, we have

$$\mathcal{A}_{T_1} \ge \int_0^{T_1}\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 + \frac{a_1}{|\gamma|^\alpha}dt =: \mathcal{D}_{T_1}(\gamma).$$

Now we consider the minimizer of $\mathcal{D}_{T_1}$ for loops with a collision at $t=0$ and T_1. The domain of $\mathcal{D}_{T_1}$ is

$$\Omega_{T_1} = \{x\in H^1([0,T_1],\mathbb{R}^N)\mid x(0)=x(T_1)=0\}.$$

Let

$$c_1 = \inf\{\mathcal{D}_{T_1}(x)\mid x\in\Omega_{T_1}\}$$

and z be the minimizer. The minimizer is attained by the collision-ejection solution, i.e. it has collisions at $t=0$ and T_1, and zero velocity at $t=\frac{T_1}{2}$. Let $z(t)$ be the minimizer. The angular momentum is zero since $z(t)$ has collisions. Then $z(t)$ moves along a line. We can regard $z(t)\in\mathbb{R}_+$. It conserves the energy

$$\frac{1}{2}\dot{z}^2 - \frac{a_1}{z^\alpha} = h_c.$$

Note that h_c is not necessarily identical with h. From

$$\frac{dz}{dt} = \sqrt{2(a_1 z^{-\alpha}+h_c)},$$

we have

$$\int_0^{z_{\max}} \frac{1}{\sqrt{2(a_1 z^{-\alpha}+h_c)}}dz = \int_0^{\frac{T_1}{2}} 1dt$$

where $z_{\max} = (a_1/(-h_c))^{1/\alpha}$. By using the substitution

$$z = \left(\frac{a_1}{-h_c}\right)^{1/\alpha} w^{1/\alpha},$$

we get

$$\frac{a_1^{\frac{1}{\alpha}}}{2^{\frac{1}{2}}\alpha} B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)(-h_c)^{-\frac{\alpha+2}{2\alpha}} = \frac{T_1}{2}.$$

From this, the value is represented by

$$h_c = -\frac{2^{\frac{\alpha}{\alpha+2}} a_1^{\frac{2}{\alpha+2}}}{T_1^{\frac{2\alpha}{\alpha+2}}\alpha^{\frac{2\alpha}{\alpha+2}}}\left(B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}}.$$

By using the integration by parts similarly as (12), we compute the exact value of h_cT_1 and c_1 as follows:

$$\begin{aligned}
h_cT_1 &= \int_0^{T_1} h_c dt = \int_0^{T_1} \frac{1}{2}\left(\frac{dz}{dt}\right)^2 - \frac{a_1}{z^\alpha}dt = a_1\left(\frac{\alpha-2}{2}\right)\int \frac{1}{z^\alpha}dt,\\
c_1 &= \int \frac{1}{2}\left(\frac{dz}{dt}\right)^2 + \frac{a_1}{z^\alpha}dt = \int -\frac{1}{2}\frac{d^2z}{dt^2}z + \frac{a_1}{z^\alpha}dt = \int -\frac{1}{2}\left(-\frac{a_1\alpha z}{z^{\alpha+2}}z\right) + \frac{a_1}{z^\alpha}dt\\
&= \frac{a_1(\alpha+2)}{2}\int \frac{1}{z^\alpha}dt = \frac{\alpha+2}{\alpha-2}h_cT_1 = \frac{2+\alpha}{2-\alpha}T_1\left(\frac{2^{\frac{1}{2}}a_1^{\frac{1}{\alpha}}}{T_1\alpha}B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}}\\
&= \frac{2+\alpha}{2-\alpha}\frac{2^{\frac{\alpha}{\alpha+2}}a_1^{\frac{2}{\alpha+2}}}{\alpha^{\frac{2\alpha}{\alpha+2}}}\left(B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}} T_1^{\frac{2-\alpha}{2+\alpha}}.
\end{aligned} \tag{16}$$

By using (15), we get

$$-hT_1 \geq \frac{2-\alpha_2}{2+\alpha_2}c_1 = \frac{2-\alpha_2}{2+\alpha_2}\frac{2+\alpha}{2-\alpha}\frac{2^{\frac{\alpha}{\alpha+2}}a_1^{\frac{2}{\alpha+2}}}{\alpha^{\frac{2\alpha}{\alpha+2}}}\left(B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}} T_1^{\frac{2-\alpha}{2+\alpha}}$$

and then

$$T_1 \geq \left(\frac{2-\alpha_2}{2+\alpha_2}\right)^{\frac{2+\alpha}{2\alpha}}\left(\frac{2+\alpha}{2-\alpha}\right)^{\frac{2+\alpha}{2\alpha}}\frac{2^{\frac{1}{2}}a_1^{\frac{1}{\alpha}}}{\alpha}B\left(\frac{1}{2}, \frac{1}{\alpha}+\frac{1}{2}\right)(-h)^{-\frac{2+\alpha}{2\alpha}}.$$

From (16), we have

$$\begin{aligned}
c_1 &= \frac{2+\alpha}{2-\alpha}\frac{2^{\frac{\alpha}{\alpha+2}}a_1^{\frac{2}{\alpha+2}}}{\alpha^{\frac{2\alpha}{\alpha+2}}}\left(B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}}T_1^{\frac{2-\alpha}{2+\alpha}}\\
&\geq \frac{2+\alpha}{2-\alpha}\frac{2^{\frac{\alpha}{\alpha+2}}a_1^{\frac{2}{\alpha+2}}}{\alpha^{\frac{2\alpha}{\alpha+2}}}\left(B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\right)^{\frac{2\alpha}{\alpha+2}}\\
&\quad\times\left(\left(\frac{2-\alpha_2}{2+\alpha_2}\right)^{\frac{2+\alpha}{2\alpha}}\left(\frac{2+\alpha}{2-\alpha}\right)^{\frac{2+\alpha}{2\alpha}}\frac{2^{\frac{1}{2}}a_1^{\frac{1}{\alpha}}}{\alpha}B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)(-h)^{-\frac{2+\alpha}{2\alpha}}\right)^{\frac{2-\alpha}{2+\alpha}}\\
&\geq \frac{2^{\frac{1}{2}}a_1^{\frac{1}{\alpha}}}{\alpha}B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\left(\frac{2-\alpha_2}{2+\alpha_2}\right)^{\frac{2-\alpha}{2\alpha}}\left(\frac{2+\alpha}{2-\alpha}\right)^{\frac{2+\alpha}{2\alpha}}(-h)^{-\frac{2-\alpha}{2\alpha}}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{T_1}\left|\frac{d\gamma}{dt}\right|^2 dt = \int_0^{T_1} h - V(\gamma)\,dt\\
&\geq \frac{\alpha_1}{2+\alpha_2}c_1\\
&\geq \frac{\alpha_1}{2+\alpha_2}\left(\frac{2-\alpha_2}{2+\alpha_2}\right)^{\frac{2-\alpha}{2\alpha}}\left(\frac{2+\alpha}{2-\alpha}\right)^{\frac{2+\alpha}{2\alpha}}\frac{2^{\frac{1}{2}}a_1^{\frac{1}{\alpha}}}{\alpha}\left(B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\right)(-h)^{-\frac{2-\alpha}{2\alpha}}\\
&=: S(a_1,\alpha,\alpha_1,\alpha_2,h).
\end{aligned}$$

Similarly, we can see

$$\frac{1}{2}\int_{T_{i-1}}^{T_i}\left|\frac{d\gamma}{dt}\right|^2 dt = \int_{T_{i-1}}^{T_i} h - V(\gamma)\,dt \geq S(a_1,\alpha,\alpha_1,\alpha_2,h).$$

Therefore, we have

$$I \geq (S(a_1,\alpha,\alpha_1,\alpha_2,h)k)\times(S(a_1,\alpha,\alpha_1,\alpha_2,h)k) = (S(a_1,\alpha,\alpha_1,\alpha_2,h))^2k^2.$$

By comparing the minimax value and the lower value for the collision paths, we get

$$(S(a_1,\alpha,\alpha_1,\alpha_2,h))^2k^2 \leq \pi^2\alpha a_2^{\frac{2}{\alpha}}\left(\frac{2-\alpha}{-2h}\right)^{\frac{2-\alpha}{\alpha}}.$$

By simplifying this inequality, we get

$$k \leq \frac{\pi b^{\frac{1}{\alpha}}\alpha^{\frac{3}{2}}(2-\alpha)^{\frac{2}{\alpha}}(2+\alpha_2)^{\frac{2+\alpha}{2\alpha}}}{2^{\frac{1}{\alpha}}\alpha_1(2+\alpha)^{\frac{2+\alpha}{2\alpha}}(2-\alpha_2)^{\frac{2-\alpha}{2\alpha}}B\left(\frac{1}{2},\frac{2+\alpha}{2\alpha}\right)}$$

and this completes the proof of our main theorem.

5. **Proof of Corollaries.** Here we prove Corollary 1 and 2.

Since

$$\alpha(2-\alpha)<1,\quad \left(\frac{2-\alpha}{2}\right)^{\frac{1}{\alpha}}<\frac{1}{2},\quad B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)>B\left(\frac{1}{2},1+\frac{1}{2}\right)=\frac{\pi}{2}$$

for $\alpha\in(1,2)$, $f(1,\alpha,\alpha,\alpha)$ is less than 1:

$$f(1,\alpha,\alpha,\alpha)=\pi\left(\alpha(2-\alpha)\right)^{\frac{1}{2}}\left(\frac{2-\alpha}{2}\right)^{\frac{1}{\alpha}}\left(B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\right)^{-1}<1.$$

This proves Corollary 1.

The gamma function is defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dx$$

and the beta function can be represented by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

It is well-known that Stirling's formula indicates

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} e^{\mu(x)}$$

where $\mu(x)$ is a function satisfying

$$0 < \mu(x) < \frac{1}{12x}.$$

Hence,

$$\begin{aligned} B\left(\frac{1}{2}, \frac{1}{\alpha} + \frac{1}{2}\right) &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\alpha}+1\right)} \\ &= \frac{\sqrt{\pi}\sqrt{2\pi}\left(\frac{1}{\alpha}+\frac{1}{2}\right)^{\frac{1}{\alpha}} e^{-\frac{1}{\alpha}-\frac{1}{2}} e^{\mu\left(\frac{1}{\alpha}+\frac{1}{2}\right)}}{\sqrt{2\pi}\left(\frac{1}{\alpha}+1\right)^{\frac{1}{\alpha}+\frac{1}{2}} e^{-\frac{1}{\alpha}-1} e^{\mu\left(\frac{1}{\alpha}+1\right)}} \\ &= \frac{\sqrt{\pi e}\left(\frac{1}{\alpha}+\frac{1}{2}\right)^{\frac{1}{\alpha}} e^{\mu\left(\frac{1}{\alpha}+\frac{1}{2}\right)-\mu\left(\frac{1}{\alpha}+1\right)}}{\left(\frac{1}{\alpha}+1\right)^{\frac{1}{\alpha}+\frac{1}{2}}}. \end{aligned}$$

By using this and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $f(1,\alpha,\alpha,\alpha)$ for $0 < \alpha < 2$ can be estimated as follows:

$$\begin{aligned} f(1,\alpha,\alpha,\alpha) &= \pi\left(\alpha(2-\alpha)\right)^{\frac{1}{2}}\left(\frac{2-\alpha}{2}\right)^{\frac{1}{\alpha}}\left(B\left(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}\right)\right)^{-1} \\ &= \pi\left(\alpha(2-\alpha)\right)^{\frac{1}{2}}\left(\frac{2-\alpha}{2}\right)^{\frac{1}{\alpha}} \frac{\left(\frac{1}{\alpha}+1\right)^{\frac{1}{\alpha}+\frac{1}{2}}}{\sqrt{\pi e}\left(\frac{1}{\alpha}+\frac{1}{2}\right)^{\frac{1}{\alpha}} e^{\mu\left(\frac{1}{\alpha}+\frac{1}{2}\right)-\mu\left(\frac{1}{\alpha}+1\right)}} \\ &= \sqrt{\frac{2\pi}{e}}\left(1-\frac{\alpha^2}{2+\alpha}\right)^{\frac{1}{\alpha}}\left(1+\frac{\alpha}{2}-\frac{\alpha^2}{2}\right)^{\frac{1}{2}} e^{\mu\left(\frac{1}{\alpha}+1\right)-\mu\left(\frac{1}{\alpha}+\frac{1}{2}\right)} \\ &\leq \sqrt{\frac{2\pi}{e}} \cdot 1 \cdot \frac{3}{2\sqrt{2}} \cdot e^{\frac{1}{12}} \\ &= 2^{-1} \cdot 3\pi^{\frac{1}{2}} e^{-\frac{5}{12}} \approx 1.75271039050823 < 2 \end{aligned}$$

This completes the proof of Corollary 2.

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